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BACHELOR THESIS

A factor model for predicting exchange rate fluctuations

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Declaration of Authorship

I, Solomiia KANTSIR, declare that this thesis titled "A factor model for predicting exchange rate fluctuations" and the work presented in it are my own. I confirm that:

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Date:

Life will overcome death, and light will overcome darkness.

Volodymyr Zelenskyy

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by Solomiia KANTSIR

Abstract

We investigate the relationship between risk and returns in the exchange rate market and propose a new statistical model for predicting currency returns using the Instrumented Principal Component Analysis (IPCA) (Kelly, Pruitt, and Su, [2019\)](#page-37-0). We show that the model with time-varying loadings and latent factors outperforms the existing factor-based strategies in-sample and out-of-sample. Specifically, the four-factor IPCA model explains up to 64% of currency returns variation, while the model with observable factors shows the performance of 57%. We have found that the IPCA factors that explain the cross-section of currency returns are global volatility, carry trade, dollar, and momentum. The results are tested in-sample and outsample and hold for individual currencies and managed portfolios.

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List of Abbreviations

- **IPCA I**nstrumented **P**rincipal **C**omponent **A**nalysis
- **PCA** Principal Component Analysis
OLS Ordinary Least Squares
- **OLS** Ordinary Least Squares
ALS Alternating Least Square
- **ALS A**lternating **L**east **S**quares
- **CAPM C**apital **A**sset **P**ricing **M**odel

Dedicated to my boyfriend, family and friends. . .

Chapter 1

Introduction

A risk-return trade-off is a central topic of asset pricing. It states that an increase in asset returns should be accompanied by an increase in risk. This concept should apply to all asset classes, including exchange rates. The goal of this work is to examine the trade-off between risks and rewards for the underexplored area of currency returns.

A common approach to investigating the relationship between risk and returns is to use the factor models. This method implies that a small set of factors can explain a cross-section of asset returns. In this way, the asset return is based on the factor risk premia and the asset's sensitivity (beta) to the particular factor. Factor models can be divided into macroeconomic, fundamental and statistical types. Macroeconomic models consider such factors as inflation, interest rates, economic growth, exchange rates, and others. Fundamental models construct observable factors based on the observable characteristics of the assets such as size, value, momentum, volatility, among others. A standard approach in the literature is the usage of the models in the spirit of Fama-French. Statistical models perceive factors as latent and estimate them using data compression techniques such as principal component analysis or time-series regressions.

There is a plethora of academic research on the observable factors in the foreign exchange market. However, it seems surprising that statistical models are not widely used for modeling currency returns. The main aim of this thesis is to fill in this gap in the literature by proposing a new statistical model for currency returns using the novel instrumented principal component analysis (IPCA). The IPCA methodology allows us to combine the strengths of both statistical and observable factor models in a unified approach.

1.1 Objectives

In this thesis, we will examine the trade-off between risk and returns in the context of the foreign exchange market and perform a latent factor analysis for a large crosssection of exchange rate returns. The IPCA methodology employed in this work allows time variation in the factor loadings based on a set of individual asset characteristics. Our goal is to present a factor model to explain the cross-section of currency returns. Moreover, we will compare the performance of a newly-found latent factor model with the existing observable factor models.

An estimation of the latent factors and betas requires an application of a data compression technique - principal components analysis (PCA). However, this approach has a major shortcoming. The PCA implies constant factor loadings (betas) that do not consider the time-varying nature of assets' exposure to the risk factors. Thus, we will apply the IPCA methodology recently developed by (Kelly, Pruitt,

and Su, [2019\)](#page-37-0). It considers the time variation in asset behavior and allows risk factor loadings to depend on observable characteristics. In the case of exchange rates, the observable characteristics will be factors such as macro-fundamental variables or currency-specific market variables. Additionally, the IPCA model examines whether a relationship between characteristics and expected return occurs due to the exposure to latent factors or if it is an anomaly (compensation without risk).

Chapter 2

Literature and related works review

2.1 Asset pricing models

The idea of factor-based investing was derived from the Capital Asset Pricing Model (CAPM) introduced in the 1960s (Lintner, [1965;](#page-37-2) Mossin, [1966;](#page-37-3) Sharpe, [1964;](#page-37-4) Treynor, [1961\)](#page-38-0). The CAPM states that only one factor, the market factor, drives the securities returns. The model has its drawbacks but is used nowadays due to its simplicity and has been a basis for further asset pricing models. Later, Ross, [1976](#page-37-5) proposed an idea of the Arbitrage Pricing Theory, where the asset return has a linear relationship and can be predicted with its expected return and a number of different macroeconomic factors capturing systematic risk. Fama and French, [1993](#page-37-6) presented a three-factor model that extend the CAPM model, adding size and value factors. Later on, the Fama French three-factor model was expanded with a momentum factor by Carhart, [1997.](#page-37-7) The question of which factors drive the assets' returns remains relevant nowadays - Fama and French, [2015](#page-37-8) presented their five-factor model and included profitability and investment factors.

2.2 Factors in the currency markets

Lustig, Roussanov, and Verdelhan, [2011](#page-37-9) identified that carry trade (slope) and dollar (level) are the global factors accounting for the 80% variation in the returns of the currency interest rate portfolio. Menkhoff et al., [2012](#page-37-10) stated that momentum strategies deliver high excess returns in the foreign exchange markets up to 10% annually. Moreover, in the later work of Menkoff et al., [2012](#page-37-11) "Carry Trades and Global Foreign Exchange Volatility", they examine that the currency volatility factor, to a large extent, explains the variance in the carry trade portfolios. They also demonstrate that liquidity risk impacts the currency excess returns, but to a lesser extent than volatility. Lustig, Roussanov, and Verdelhan, [2012](#page-37-12) have proposed a new currency investment strategy - "dollar carry trade." Following this strategy, the investor should go long on the foreign currencies and short on the dollar when foreign short-term interest rates exceed the U.S. short-term interest rates and, on the contrary, long dollar currency and short all foreign ones. Also, Della Corte et al., [2021](#page-37-13) observed that a country's sovereign risk (risk of its default on debt) is a prominent source of risk in currency markets. An increase in a country's sovereign risk leads to the depreciation of its currency and an increase in volatility. We will use the majority of the risk factors mentioned above in our analysis as the observable factors. Specifically, we will choose the dollar, carry trade, momentum, volatility, and dollar carry trade factors as our benchmarks.

2.3 Instrumented Principal Component Analysis

An important work for our research is the paper "Characteristics are covariances: A unified model of risk and return" by Kelly, Pruitt, and Su, [2019.](#page-37-0) They propose a novel approach to analyzing the cross-section of returns - an Instrumented Principal Component Analysis (IPCA). Büchner and Kelly, [2022,](#page-37-14) in their work "A factor model for option returns" also apply the IPCA technique to find the latent risk factors in option returns. The IPCA model allows for time-varying loadings and latent factors instrumented by some observable characteristics. This approach rules out the problems of the time-varying nature of the risk exposures and lack of the full knowledge of the returns cross-section. The IPCA approach is being used in many other academic papers. For instance, Bianchi and Babiak, [2021](#page-37-15) and Kelly, Palhares, and Pruitt, [2022](#page-37-16) apply the IPCA methodology to examine the risk-return trade-off in the cryptocurrency and corporate bond markets. The IPCA technique is especially relevant for the exchange rates market and is central in our investigation of the latent factors of the currency returns. Moreover, this thesis is the first work where the IPCA is applied to investigate exchange rate fluctuations.

Chapter 3

Data overview

3.1 Exchange rates and characteristics

The characteristics and factors data are obtained from Thompson Reuters Datastream. The sample includes monthly data for the exchange rates of 37 countries from January 1992 to September 2018. The countries are Australia, Austria, Belgium, Canada, Chile, Czech Republic, Denmark, Eurozone (A19), Finland, France, Germany, Greece, Hungary, India, Indonesia, Ireland, Italy, Japan, Malaysia, Mexico, Netherlands, New Zealand, Norway, Philippines, Poland, Portugal, Saudi Arabia, Singapore, South Africa, South Korea, Spain, Sweden, Switzerland, Taiwan, Thailand, Turkey, United Kingdom. We consider exchange rates against the U.S. dollar.

[Table 3.1](#page-16-0) reports the summary statistics (mean, standard deviation, median, skewness, kurtosis, Sharpe ratio) of the exchange rate for each country and the number of observations. Some countries do not have data for some periods and therefore enter the sample later. Also, we eliminate the EU members in the year of euro adoption and further. [Table A.1](#page-32-1) represents the dates range of available data for each country and number of observations.

Moreover, for each country, we have 16 characteristics: harmonized unemployment rate (*HUR*), the growth of consumer price index (*CPI*), money supply (*M3*), industrial production growth (*INDPROD*), balance of payments (*BOP*), government budget deficit (*GOVDEF*), producers price index (*PPI*), business confidence index (*BCI*), consumer confidence index (*CCI*), momentum defined as the average exchange rate growth over the last 1, 3, 6 and 12 months (*FXMOM1, FXMOM3, FXMOM6, FX-MOM12*), volatility defined as a sum of squared daily exchange rate returns (*FXVOL*), a difference between interest rates on long and short term securities (*TERMSPREAD*), one-month interest rate (*INTRATE*). There are 16,341 country-month observations for each characteristic and FX before data tidying. The characteristics' values are expressed as the difference between the U.S. and foreign countries' values.

We tidy the data by deleting some missing observations and performing the transformation. In particular, we carry out rank normalization — replace observations with their fractional rank, the currency rank divided by the number of nonmissing observations. Then we subtract 0.5 from each observation to scale values into the [0.5, +0.5] interval. In this way, we make our data less sensitive to the outliers while focusing on ordering rather than magnitude. After removing missing values, there are 8,162 for FX and characteristics.

[Figure 3.1](#page-17-1) shows the correlation between all the characteristics. There is no strong correlation between variables, except for a negative correlation of -0.77 between the interest rate (*IntRate*) and a difference between interest rates on long and short-term securities (*TERMSPREAD*). This correlation is expected to a tight dependance between the short- and long-term rates. Intuitively, if the interest rate (a

TABLE 3.1: FX summary statistics

This table reports the summary statistics of the exchange rate for each country on the monthly basis. In particular, the table shows mean, standard deviation, median, skewness, kurtosis, Sharpe ratio and the number of observations. Mean, standard deviation and median are expressed as a percentage.

	FX	HUR	CPI	M3	INDPROD	BOP	GOVDEF	PPI	BCI	CCI	FXMOM1	FXMOM3	FXMOM6	FXMOM12	FXVOL	TERMSPREAD	IntRate
FX	1.00	-0.04	0.13	0.06	-0.04	0.06	0.02	0.10	0.01	0.02	0.05	0.05	0.04	0.03	-0.01	-0.06	0.10
HUR	-0.04	1.00	-0.27	-0.04	0.02	0.09	-0.51	0.15	0.10	-0.40	-0.04	-0.08	-0.11	-0.12	-0.11	0.31	-0.23
CPI	0.13	-0.27	1.00	0.08	-0.01	-0.12	0.03	0.48	-0.07	-0.00	0.11	0.13	0.10	-0.05	0.13	-0.26	0.45
M ₃	0.06	-0.04	0.08	1.00	-0.00	-0.02	0.00	0.05	0.00	0.05	-0.02	-0.02	0.01	0.06	-0.00	-0.02	0.11
INDPROD	-0.04	0.02	-0.01	-0.00	1.00	0.01	-0.00	0.00	0.06	0.01	-0.03	0.00	-0.02	-0.04	-0.03	0.02	-0.04
BOP	0.06	0.09	-0.12	-0.02	0.01	1.00	0.12	-0.02	0.14	-0.03	0.06	0.08	0.05	0.03	-0.05	0.13	-0.23
GOVDEF	0.02	-0.51	0.03	0.00	-0.00	0.12	1.00	-0.17	-0.11	0.35	0.01	0.03	0.04	0.04	0.09	-0.20	-0.00
PPI	0.10	0.15	0.48	0.05	0.00	-0.02	-0.17	1.00	0.11	-0.19	0.06	-0.02	-0.09	-0.29	0.11	0.02	0.17
BCI	0.01	0.10	-0.07	0.00	0.06	0.14	-0.11	0.11	1.00	0.20	0.02	0.05	0.06	0.05	0.00	0.29	-0.25
CCI	0.02	-0.40	-0.00	0.05	0.01	-0.03	0.35	-0.19	0.20	1.00	0.04	0.14	0.22	0.26	0.05	-0.04	0.09
FXMOM1	0.05	-0.04	0.11	-0.02	-0.03	0.06	0.01	0.06	0.02	0.04	1.00	0.58	0.41	0.30	-0.14	-0.07	0.09
FXMOM3	0.05	-0.08	0.13	-0.02	0.00	0.08	0.03	-0.02	0.05	0.14	0.58	1.00	0.72	0.52	-0.12	-0.10	0.15
FXMOM6	0.04	-0.11	0.10	0.01	-0.02	0.05	0.04	-0.09	0.06	0.22	0.41	0.72	1.00	0.71	-0.10	-0.11	0.21
FXMOM12	0.03	-0.12	-0.05	0.06	-0.04	0.03	0.04	-0.29	0.05	0.26	0.30	0.52	0.71	1.00	-0.09	-0.07	0.25
FXVOL	-0.01	-0.11	0.13	-0.00	-0.03	-0.05	0.09	0.11	0.00	0.05	-0.14	-0.12	-0.10	-0.09	1.00 ₁	-0.09	0.18
MSPREAD	-0.06	0.31	-0.26	-0.02	0.02	0.13	-0.20	0.02	0.29	-0.04	-0.07	-0.10	-0.11	-0.07	-0.09	1.00	-0.77
IntRate	0.10	-0.23	0.45	0.11	-0.04	-0.23	-0.00	0.17	-0.25	0.09	0.09	0.15	0.21	0.25	0.18	-0.77	1.00

FIGURE 3.1: Characteristics correlation matrix This figure shows the pairwise correlation between characteristics.

short-term one) increases, the difference between long and short-term interest rates decreases.

3.2 Observable risk factors

We analyze five observable factors — dollar, carry trade, momentum, volatility, and dollar carry trade. These factors were chosen because they have been shown to accurately explain the variation in the currency returns in previous academic works (e.g., Lustig, Roussanov, and Verdelhan, [2011,](#page-37-9) Menkoff et al., [2012,](#page-37-11) Della Corte et al., [2021\)](#page-37-13).

The dollar-factor strategy is constructed as an equally-weighted portfolio where we invest an equal amount of money into each currency. As a result, we get the average excess return on all foreign currency. The carry trade factor is constructed as a return of high-interest-rate currencies minus the return of low-interest-rate currencies. We construct the momentum factor by building a portfolio that goes long (short) on currencies that have recently yielded positive (negative) returns. The volatility factor contains returns on a portfolio that goes long on highly volatile currencies and short on low volatile ones. Finally, the dollar carry trade factor is constructed as a strategy of going long (short) on the foreign currencies and short (long) on the dollar when foreign short-term interest rates exceed (are less than) the U.S. short-term interest rates (Lustig, Roussanov, and Verdelhan, [2012\)](#page-37-12).

[Table 3.2](#page-18-1) shows the descriptive statistics of the factors data. The factor data is from January 1992 to September 2018. The values stand for the percentage of return change per 1\$ of investment in a particular factor. [Figure 3.2](#page-18-0) summarizes the correlation between observable risk factors. We can observe strong positive relationship between volatility and dollar factors.

TABLE 3.2: Observable factors summary statistics

This table reports the summary statistics of the five observable factors on the yearly basis. The factor data is from January 1992 to September 2018. The values stand for the percentage of return change per 1\$ of investment in a particular factor.

	dollar	carry	mom	vol	val
dollar	1.00	0.32	-0.08	0.76	-0.12
carry	0.32	1.00	-0.11	0.46	-0.06
mom	-0.08	-0.11	1.00	-0.17	0.09
vol	0.76	0.46	-0.17	1.00	-0.09
val	-0.12	-0.06	0.09	-0.09	1.00

FIGURE 3.2: Factors correlation matrix

This figure shows the pairwise correlation between observable risk factors.

Chapter 4

Methodology

4.1 Instrumented Principal Component Analysis

In order to estimate latent factors and factor loadings, we will imply the approach of Instrumented Principal Component Analysis developed by (Kelly, Pruitt, and Su, [2019\)](#page-37-0). The IPCA method determines latent factors and factor loadings based on the cross-section of the currency returns. It also allows loadings to be time-varying and dependent on the panel of the country-specific characteristics. An IPCA model specification for excess returns $r_{i,t+1}$ for currency *i* at time $t + 1$ is:

$$
r_{i,t+1} = \alpha_{i,t} + \beta_{i,t} f_{t+1} + \epsilon_{i,t+1}
$$
\n
$$
\alpha_{i,t} = z'_{i,t} \Gamma_{\alpha} + \nu_{\alpha,i,t}, \quad \beta_{i,t} = z'_{i,t} \Gamma_{\beta} + \nu_{\beta,i,t},
$$
\n(4.1)

where f_{t+1} is is a *K*-vector of latent factors with $\beta_{i,t}$ being dynamic factor loadings, an exposure to systematic risk factors. These loadings $\beta_{i,t}$ are allowed to be timevarying and are a linear function of observable asset characteristics z['] *i*,*t* for country *i* at time *t*, *Lx*1 vector. Γ_{β} is a parameter matrix that defines a mapping from characteristics to risk factors - the matrix contains the weights of of characteristics. The intercepts $\alpha_{i,t}$ are also allowed to be time-varying and depending on the observable asset characteristics *z* ′ *i*,*t* . The error term *νβ*,*i*,*^t* is assumed to be zero-mean and orthogonal to risk factors. Time-varying intercepts and betas are the key concepts of IPCA that differ it from the PCA model where loadings are constant.

If our model imply $\alpha_{i,t} = 0$ that means that factors entirely explain the variation of expected returns and proxy for systematic risk exposures. Otherwise, if intercepts $\alpha_{i,t}$ are non-zero, then expected returns have intercepts that depend on stock characteristics. Thus, we can deduce there are excess returns that do not align with systematic risk exposure. In other words, there is a so-called "anomaly".

We build the null hypothesis H_0 : $\Gamma_\alpha = 0$ that characteristics do not proxy for intercepts $\alpha_{i,t}$ by restricting Γ_α to zero. It indicates that systematic factors entirely drive the expected returns. An alternative hypothesis $H_1 : \Gamma_\alpha \neq 0$ states that characteristics impact the excess returns through intercepts, not risk factors. That means the currency compensation is determined not by the exposure to the systematic factors. Intercepts *αi*,*^t* are estimated by finding a linear combination of characteristics that most accurately describe the expected returns while controlling for the role of characteristics in factor risk exposure.

The IPCA technique plays a crucial role in our analysis. First of all, instrumenting the latent factors with pre-specified characteristics enables additional data to build a factor model of the currency returns. It improves the accuracy of our estimation and the performance of the model. Also, the IPCA incorporates the time-varying loadings that bring more explanatory power to the factor model of currency returns.

4.2 IPCA estimation

The IPCA approach reduces the dimensionality by eliminating correlated, noisy and uninformative characteristics. It selects a few linear combinations of characteristics that are the most descriptive about the returns.

From the equation [4.1](#page-19-2) we derive its vector form:

$$
r_{i,t+1} = Z_t \Gamma_\beta f_{t+1} + \epsilon_{t+1}^* \tag{4.2}
$$

, where $r_{i,t+1}$ is a vector of currency excess returns, Z_t consists of characteristics for each country at time *t*, and ϵ_{t+1}^* is a composite error:

$$
\epsilon_{i,t+1}^* = \epsilon_{i,t+1} + \nu_{\alpha,i,t} + \nu_{\beta,i,t} f_{t+1}
$$
\n(4.3)

The aim is to find such values of the matrix of factors' loadings Γ*^β* and latent factors f_{t+1} to minimize the sum of squared composite errors of the model. Following this approach, f_{t+1} and Γ_β should satisfy the first-order condition. There is no a closed-form solution, and Kelly, Pruitt, and Su, [2019](#page-37-0) propose the alternating least squares method to calculate the above mentioned system of the first-order conditions for latent factors and Γ*β*.

We construct the characteristics-based portfolio returns:

$$
x_{t+1} = \frac{Z_t' r_{t+1}}{N_{t+1}}
$$
\n(4.4)

, where Z'_t N_t is the *NxL* matrix of characteristics for each country at time *t*, N_{t+1} is the number of non-missing observation each month used for data normalization. As a result, *xt*+¹ is a *Lx*1 vector of weighted average currency returns with weights determined by the characteristic value. We interpret the portfolio the following way - if the characteristic weight is negative we short (sell) the currency and if positive we long (buy) the currency.

The next step is to initialize the Γ_β as the first eigenvectors of the second moment matrix $\sum_t x_t x_t'$ \mathbf{z}'_t . Due to the time-varying structure of $\mathbf{Z}'_t \mathbf{Z}_t$, this is only an approximate solution, but it is enough to be an initial guess as a starting point. Afterwards, using the latest value of Γ*^β* we evaluate the latent factors returns using the OLS (Ordinary Least Squares). Then, using the latest estimate of latent factors, we evaluate the Γ*^β* via the OLS. The algorithm performs until the difference between the estimates is smaller than 10^{-6} . To sum up, we iteratively try to find the best estimate for both Γ_β and f_{t+1} via the OLS.

4.3 Asset Pricing Tests

We calculate the IPCA model for different choices of factors. For each of the K-factor model, we construct both restricted ($\Gamma_{\alpha} = 0$) and unrestricted ($\Gamma_{\alpha} \neq 0$) versions.

A good model should accurately describe the variation in excess returns and describe risk compensation. In order to asses the IPCA model performance we estimate both total panel R_{total}^2 and predictive R_{pred}^2 .

4.3.1 Total *R* 2

$$
R_{total}^{2} = 1 - \frac{\sum_{i,t} (r_{i,t+1} - z_{i,t}' (\hat{\Gamma}_{\alpha} + \hat{\Gamma}_{\beta} \hat{f}_{t+1}))}{\sum_{i,t} r_{i,t+1}^{2}}
$$
(4.5)

The total R^2 is the fraction of variance in returns $r_{i,t+1}$ described by the estimated latent factors \hat{f}_{t+1} , estimated dynamic loadings $\hat{\beta}_{i,t+1}$ and alphas $\hat{\alpha}_{i,t}$ in the unrestricted IPCA model.

4.3.2 Predictive *R* 2

$$
R_{pred}^{2} = 1 - \frac{\sum_{i,t} (r_{i,t+1} - z'_{i,t} (\hat{\Gamma}_{\alpha} + \hat{\Gamma}_{\beta} \hat{\lambda}))}{\sum_{i,t} r_{i,t+1}^{2}},
$$
(4.6)

The predictive R^2 measures how accurately the expected returns variation is explained by the model's conditional expected returns. $\hat{\lambda}$ is an unconditional mean of factors. This metric estimate how well the returns are explained through instruments, eliminating the effect of risk prices dynamics.

4.4 Hypotheses testing

We will perform several bootstrap Wald tests to assess the performance of the IPCA model:

4.4.1 Testing zero alpha condition Γ*^α* = 0

As we have mentioned above, we build two hypotheses. The null hypothesis *H*⁰ : $\Gamma_\alpha = 0$ states that characteristics describe the returns through systematic risk factors. The alternative hypothesis $H_1: \Gamma_\alpha \neq 0$ states that there are some anomalies in asset pricing that are not connected to the risk factors. The bootstrap test in this case is performed by measuring the distance between unrestricted alpha and zero. That is when unrestricted alpha estimates are too far from zero it means that unrestricted model (with non-zero alphas) is the true one. The procedure is outlined in Kelly, Pruitt, and Su, [2019,](#page-37-0) and generates pseudo-samples under the restricted model hypothesis. In particular, we compute the Wald-type statistics:

$$
W_{\alpha} = \hat{\Gamma}_{\alpha}' \hat{\Gamma}_{\alpha} \tag{4.7}
$$

Then, we evaluate the unrestricted model and get the estimates of $\hat{\Gamma}_{\alpha}$, $\hat{\Gamma}_{\alpha}$ matrix with characteristics weights, and latent factors $\{\hat{f}_t\}_{t=1}^T$. Next, we generate the bootstrap samples for managed portfolio returns x_t for $b = 1, ..., 1000$ as:

$$
\widetilde{x}_{t+1}^b = (Z_t' Z_t) \widehat{\Gamma}_{\beta} \widehat{f}_{t+1} + \widetilde{d}_{t+1}^b, \quad \widetilde{d}_{t+1}^b = q_{1,t+1}^b \widehat{d}_{q_{2,t+1}^b}.
$$
\n(4.8)

We draw the residual $\hat{d}_{q_{2,t+1}^b}$ in a random time index from our possible dates and multiply it by a random Student t-variable $q_{1,t+1}^b$. Having that bootstrap sample, we re-estimate the unrestricted model and construct the bootstrapped Wald-type statistics: ′

$$
\hat{W}^b_\alpha = \hat{\Gamma}^b_\alpha \hat{\Gamma}^b_\alpha \tag{4.9}
$$

Finally, we calculate p-value of the null hypothesis as the fraction of bootstrapped Wald-type statistics that exceeds the actual Wald-type statistics.

4.4.2 Testing characteristics significance

We will be testing the significance of each separate characteristic and its contribution to the factor loadings *βi*,*^t* . For this test, we present the Γ*^β* as the matrix with *Kx*1 vectors *γβ*,*^l* with weights of characteristic *l* on each of the *K* factors.

$$
\Gamma_{\beta} = [\gamma_{\beta,1}, ..., \gamma_{\beta,L}]'.
$$
\n(4.10)

The hypotheses for this test are:

$$
H_0: \Gamma_{\beta} = [\gamma_{\beta,1}, ..., \gamma_{\beta,l-1}, 0_{KxL}, \gamma_{\beta,l+1}, ..., \gamma_{\beta,L}]; \quad H_1: \Gamma_{\beta} = [\gamma_{\beta,1}, ..., \gamma_{\beta,L}] \tag{4.11}
$$

*H*⁰ states that a weight of a characteristic *l* in the matrix Γ*^β* is equal to zero and does not impact any of the factors, while *H*¹ claims non-zero contribution of the *l th* characteristic. We aim to estimate the distance between zero and *γβ*,*^l* - if the distance is large enough, then we can claim the *l th* characteristic makes significant contribution to the latent factors' loadings. We construct a Wald-type statistics of the actual IPCA model for each characteristic as:

$$
W_{\beta,l} = \gamma_{\beta,l}^{\prime} \gamma_{\beta,l}^{\prime}
$$
 (4.12)

We estimate our alternative model, and then for each bootstrap draw $b = 1,..,1000$ we re-sample the characteristics-managed portfolio returns with residuals \hat{d}_t^b determined in [subsection 4.4.1.](#page-21-3)

Chapter 5

Business Importance

Portfolio managers have always sought a risk-return trade-off, and factor-based investment has become a valuable risk management tool. Factor-based investing helps construct a portfolio that yields higher returns and reduces volatility over the long term. Moreover, it helps investment professionals understand drivers of portfolio risk and diversify their currency portfolio by investing in different factors. In particular, instead of investing in individual currencies, building a currency factor portfolio helps average out the country-specific risk (Cerrato, Li, and Zhang, [2021\)](#page-37-17). Portfolio managers can use the latent factors discovered in this thesis to build a diversified, low-volatile, and profitable portfolio.

What is also important, investors or portfolio mangers can use the IPCA model to predict the exchange rate fluctuations before buying particular currencies. For instance, with the obtained in this thesis latent factors, one can use up-to-date macroeconomic characteristics as a proxy for factors' loadings and predict the currency exchange rate in a following month. If the expected fluctuation is negative (positive), then investors sells (buys) the currency. Furthermore, the IPCA model developed further in this thesis proposes a convenient way for portfolio managers to evaluate the impact of new characteristics while controlling for those used in this research. When a new characteristic arises, it can be included in the IPCA model, and its marginal contribution to the factor loadings will be estimated (Kelly, Pruitt, and Su, [2019\)](#page-37-0).

Chapter 6

Results

6.1 IPCA model performance

We estimate the IPCA model for the various number of factors $K = 1, 2, ..., 7$. We consider two metrics to assess the model performance — a total R_{tot}^2 and predictive R_{pred}^2 . R_{tot}^2 measures how accurately the model describes the variance in currency returns. R_{pred}^2 measures how well the returns are explained through instruments, eliminating the effect of risk price dynamics. We apply these metrics to evaluate the performance of both individual assets and characteristics-weighted portfolios. Moreover, we consider the Wald p-value statistics to test the hypothesis $H_0: \Gamma_\alpha = 0$ against $H_1: \Gamma_\alpha \neq 0$.

[Table 6.1](#page-24-2) shows the R_{tot}^2 and predictive R_{pred}^2 for different specifications of the IPCA models - restricted and unrestricted configurations for individual assets and managed portfolios.

TABLE 6.1: IPCA model performance

This table shows the total R_{tot}^2 and predictive R_{pred}^2 for the IPCA models with $K = 1, 2, ..., 7$. Panel A reports the results for individual currencies, Panel B - for managed portfolios. Panel C shows the p-value for $H_0: \Gamma_\alpha = 0$.

Panel A of [Table 6.1](#page-24-2) shows the results for individual asset models. The restricted IPCA model $\Gamma_{\alpha} = 0$ with only one latent factor has a total R_{tot}^2 of 48,05%, meaning that one factor explains 48,05% of the variation in currency returns $r_{i,t+1}$. The predictive R_{pred}^2 for this model is 0,15%. Allowing the unrestricted alphas in the model

increases the total R_{tot}^2 by 0.67% to 48.72%, and the predictive R_{pred}^2 to 0,91%. That means the model betas explain the majority of variation, compared to alphas.

When increasing the number of factors in the IPCA model, the total R_{tot}^2 increases for both restricted and unrestricted models. At $k = 4$, R_{tot}^2 is 64,12% for the restricted model and 64,36% for the unrestricted one. However, the predictive R_{pred}^2 is at its highest for $k = 4$ for restricted and unrestricted models, 0,56% and 0,84%, respectively, and with $k > 4$ the metric starts to decrease.

Panel B of [Table 6.1](#page-24-2) reports the performance results of the managed portfolio models. In this case, our dependent variable is *x^t* , a vector of returns on 17 portfolios, corresponding to 16 characteristics plus constant. The performance of portfolios is significantly better due to the fact that portfolios reduce the idiosyncratic variation (Kelly, Pruitt, and Su, [2019\)](#page-37-0). The restricted IPCA model $\Gamma_{\alpha} = 0$ for portfolios with only one latent factor has a total R_{tot}^2 of 90,30%, meaning that characteristics explain almost all the variation in currency returns. The predictive R_{pred}^2 for this model is 0,28%. The one-factor unrestricted model $\Gamma_\alpha \neq 0$ has a total R_{tot}^2 of 90,71% and a predictive R_{pred}^2 of 0,73%. When increasing the number of factors to $k = 4$, we observe a significant increase by 6,67% in the total R_{tot}^2 of a restricted model, and by 6,27% of an unrestricted model. The total R_{tot}^2 is equalized for both restricted and unrestricted models with $k = 5$, meaning that alpha does not explain any returns variation in such model configuration. Similar to the individual assets case, the predictive R_{pred}^2 for the managed portfolios peaks at $k = 4$.

Panel C of [Table 6.1](#page-24-2) shows the bootstrapped p-values for the test of $\Gamma_{\alpha} = 0$. We fail to reject H_0 : $\Gamma_\alpha = 0$ at 5% significance level in models with all factors configurations $k = 1, 2, ..., 7$. The model with $k = 3$ has the p-value for $\Gamma_{\alpha} = 0$ of 22.5%. With $k = 4$, the p-value for $\Gamma_{\alpha} = 0$ rises to 77%, but with $k = 5$ p-value slightly decreases to 69.2%. We conclude that the IPCA explains almost all of the variation in the currency expected returns associated with the characteristics.

We observe that the models with managed portfolios perform better than those with individual assets because of the noise reduction. Moreover, for both cases the IPCA models (both restricted and unrestricted) with $k = 4$ have the highest predictive R_{pred}^2 , meaning that the expected return variation is well explained even with constant estimated risk prices. To sum up, we see that at $k = 4$ almost all of the variation is associated with characteristics, not alphas. Moreover, according to the bootstrap test, we can not reject $H_0: \Gamma_\alpha = 0$ for all models. We will use the restricted model with four latent factors as a benchmark in our further analysis due to its best performance among other model specifications.

6.2 Comparison with the observable factors

We compare the IPCA to the model with pre-specified observable factors. We consider models with $K = 1, 2, ..., 5$ observable factors. The first model includes only one factor — carry trade (*C*), the second one extends with the dollar factor (*CD*). The $K = 3$ model includes carry trade, dollar and momentum (*CDM*). The $K = 4$ model extends with volatility factor (*CDMV*), and the $K = 5$ model extends with dollar carry trade factor (*CDMVT*).

We consider two types of models with observable factors — with instruments and without instruments. The observable factors model with instruments is specified similarly to the IPCA model but with the pre-specified factors instead of the latent ones. The IPCA model with observable factors has dynamic betas as a function of characteristics. The other one, the model with no instruments, is estimated as a time-series regression with constant betas. For all models, we restrict intercepts to zero Γ*^α* = 0.

[Table 6.2](#page-26-0) represents the comparison between the restricted IPCA model for *K* = 1, 2, ..., 5 latent factors (Panel A), the model with observable factors instrumented with characteristics (Panel B), and the model with static loadings corresponding to a panel regression (Panel C).

Panel B: Observable Factors - With instruments

TABLE 6.2: IPCA and observable factors models

This table compares the performance of the IPCA model and observable factors models. Panel A represents the results for the IPCA. Panel B shows the results for the observable factors model with instrumented loadings. Panel C summarizes the results for panel regression with observable factors.

From the perspective of the individual assets, the IPCA model shows better performance than the model with instrumented observable factors. The latter, a model with carry trade factor *C*, explains 12% of the variation in currency returns, while the one-factor IPCA explains almost half of the variation. When adding more observable factors to the model, the total R_{tot}^2 increases, and the difference between the IPCA gets smaller, but it still exists. At $K = 4$, the total R_{tot}^2 of the IPCA is 64,12% versus R_{tot}^2 of *CDMV* is 57.80%. The predictive R_{pred}^2 is slightly bigger for *CDMV* than for the four-factor IPCA — 0,56% and 0,58%, respectively. For managed portfolios, the explanatory power of the instrumented observable factors models is smaller than the IPCA. For instance, the total R_{tot}^2 of the four-factor CDMV model is 95,27% compared to the IPCA's R_{tot}^2 of 96,97%.

When comparing the IPCA model to the observable factors model with static loadings (panel regression), we can observe that the IPCA still outperforms. For both individual currencies and managed portfolios, restricting the loadings to be static results in a decrease in the total R_{tot}^2 compared to both IPCA and observable factors model with dynamic loadings. For example, the total R_{tot}^2 of the $CDMV$ model with

static loadings is 53,87% (individual assets), while R_{tot}^2 of four-factor IPCA is 64,12%. The predictive R_{pred}^2 is better for some choices of *k* of the panel regression, but for $k > 4$ the situation changes, and R_{pred}^2 of the IPCA becomes higher.

To sum up, the IPCA model with latent factors shows the best performance among other models with the pre-specified observable factors. It explains up to 97% of the heterogeneity in average returns. Furthermore, we have proven that the model with dynamic loadings instrumented with characteristics outperforms the model with static loadings.

6.3 Out-of-sample performance

We have already demonstrated the superior in-sample performance of the IPCA model. For this estimation, we have used a full panel of currency returns. The next step is to analyze the out-of-sample fits of the IPCA and check the validity of our model.

We divide our data into two sets — 50% of the data goes to the train set, and 50% goes to the test set. We perform recursive forecasts on our dataset starting from May 2005, following the procedure by Kelly, Pruitt, and Su, [2019.](#page-37-0) We estimate the performance of the out-of-sample model with the total R_{tot}^2 and predictive R_{pred}^2 statistics both for individual currencies and managed portfolios. Moreover, we perform the out-of-sample estimation for the models with observable factors, both with dynamic and static loadings.

[Table 6.3](#page-27-1) shows the out-of-sample performance of the models.

			Panel A: IPCA			
				4	h	h
	56,46%	60,57% 65,81%		67,90%	69,92%	71,98%
R_{total}^2 R_{pred}^2	0.42%	0.41% 0.55%		$0,60\%$	0.59%	0.93%
	92,63%	94,60% 96,73%		97,21%	97,64%	98,11%
$R^2_{total,x}$ $R^2_{pred,x}$	$0,65\%$	0.64% 0.63%		0.61%	0.65%	0.83%

Panel B: Observable Factors - With instruments

TABLE 6.3: Out-of-sample performance

This table reports the out-of-sample performance of the IPCA model (Panel A), observable factors model with instrumented loadings (Panel B), and panel regression with observable factors (Panel C).

The IPCA model with latent factors demonstrates a superior performance even out-of-sample and shows a similar behavior as in-sample. First of all, the IPCA model explains more variation than the model with instrumented observable factors. For instance, the total R_{tot}^2 for the out-of-sample baseline IPCA model specification with $k = 4$ is 67,90% for individual assets, and it increases to 97,21% for managed portfolios. Meanwhile, the out-of-sample four-factor model *CDMV* with instrumented observable factors explains 63,10% of the currency returns variation for individual assets and 96,08% for the managed portfolios. The predictive R_{pred}^2 is slightly lower for the IPCA model compared to the observable factors model for all choices of *k*, except for our benchmark IPCA model with $k = 4$.

Furthermore, the IPCA model significantly outperforms the observable factors model without instruments. For comparison, the *CDMV* model with static loadings has R_{tot}^2 of 59,26% for individual assets and 96,41% for managed portfolios. For *CDM*, *CDMV* and *CDMVT* models, in managed portfolio case, the performance of the non-instrumented observable factors is slightly better than instrumented observable factors. However, the predictive R_{pred}^2 is higher for all choices of *k* for the instrumented observable factors models.

To sum up, the IPCA model with latent factors outperforms the model with observable factors even out-of-sample. The IPCA shows superior results out-of-sample and explains up to 98% of variation in the currency returns.

6.4 Factors and characteristics

In this section, we will evaluate the marginal significance of each characteristic in our IPCA model. Furthermore, we will provide an approximate economic interpretation of the latent factors. We consider the results for our baseline model specification a four-factor restricted IPCA model.

6.4.1 Testing characteristics significance

The procedure for detecting the significant characteristics is outlined in Kelly, Pruitt, and Su, [2019](#page-37-0) and described in [subsection 4.4.2.](#page-22-0) It is a bootstrap test where we test whether the whole row, corresponding to a particular characteristic in the weights matrix Γ*β*, is equal to zero. We test the statistical significance of the characteristics at the 5% significance level.

[Table 6.4](#page-29-0) reports the characteristics p-values obtained by performing the bootstrap test.

We can conclude that out of 16 characteristics (plus constant), only six are statistically significant and have closest-to-zero bootstrapped p-values (< 0.05) . Specifically, *FXMOM*1, *FXMOM*3, *FXMOM*6, *FXVOL*, *interest rate*, and *constant* are relevant and contribute to explaining the currency returns' variance.

6.4.2 Interpretation of factors

We will analyze the characteristics' weights in Γ*^β* matrix, and thus provide an approximate interpretation of the IPCA factors. Moreover, we will take into consideration the correlation between the IPCA factors and pre-specified observable factors (dollar, carry trade, momentum, volatility, and dollar carry trade).

[Table 6.5](#page-29-1) represents the characteristics' coefficients for each factor in the matrix. [Appendix B](#page-33-0) provides summary statistics (mean, standard deviation, Sharpe ratio,

TABLE 6.4: IPCA characteristics significance

This table reports the p-value of each characteristic in the restricted IPCA model with $K = 4$. A characteristic's significance is measured by performing the bootstrap test.

kurtosis, skewness) for each IPCA factor. [Appendix C](#page-34-0) shows the correlation matrix for latent and observable factors.

TABLE 6.5: Characteristics' weights in Γ*^β*

The table shows the characteristics' weights for each latent factor in the Γ*^β* matrix.

The loadings on Factor 1 are associated with an average exchange rate growth over the last six months, interest rate, and volatility. Currencies with higher average exchange rate growth tend to load negatively on Factor 1, while the currencies with lower growth load positively on Factor 1. Meanwhile, currencies with higher interest rates and volatility load positively on Factor 1 and vice versa. It aligns with the positive correlation of Factor 1 with pre-specified factors carry trade (42% correlated) and volatility (49% correlated) (See [Appendix C\)](#page-34-0). Factor 1 is the global risk factor that can be proxied by **global volatility**. The reason is that the factor strongly correlates with the volatility and the carry trade portfolios. This strong correlation with the two observable factors is intuitive since the global volatility factors have been proposed to explain the carry trade returns.

Exposure to Factor 2 is defined mostly by the 1-month momentum, interest rate, and volatility. The expected annual return on this factor is 2% with an annual Shape ratio of 0.61 (See [Appendix B\)](#page-33-0). Factor 2 negatively correlates with the momentum factor and positively correlates with volatility and carry trade factors. Factor 2 can be interpreted as the **carry trade strategy**. The reason is twofold. First, Factor 2 has the strongest correlation with the carry trade returns. Second, Factor 2 has a positive and large coefficient for the interest rate differential.

Note that the negative coefficients (See [Figure 3.2\)](#page-18-0) of some momentum characteristics for the first and second IPCA factors could be explained by the negative correlation of the momentum strategy with the volatility and carry trade returns. Indeed, as the first two latent factors proxy for the global volatility or carry trade risks, they load negatively on some past performance characteristics.

Loadings on Factor 3 predominantly correspond to constant and interest rate. That means all currencies share a common component defining their fluctuation. Moreover, Factor 3 has a strong negative correlation of 78% with the dollar factor. Thus, we can determine Factor 3 as the **dollar factor**. It has an average annualized return of 3% and Sharpe ratio of 0.31.

Factor 4 has loadings mostly attributable to the 3-month average currency growth and consumer price index. This latent factor also has a positive correlation of 41% with the momentum factor. As a result, Factor 4 can be interpreted as a **momentum factor**. This factor brings 3% of the annual return.

Chapter 7

Conclusion

In this thesis, we apply the Instrumented Principal Component Analysis technique, developed by (Kelly, Pruitt, and Su, [2019\)](#page-37-0), to investigate the risk-return trade-off within the foreign exchange market and predict the currency fluctuations. We consider a cross-section of 37 countries and 16 country-specific macroeconomic characteristics. We build a model with latent factors and time-varying betas instrumented with these characteristics. This approach allows capturing the dynamic structure of risk factors, thus improving the model's ability to explain the variation of the currency returns accurately. Moreover, we compare the performance of the IPCA model with the set of pre-specified tradable factors in-sample and out-of-sample and perform additional robustness checks. The key conclusions of this research are threefold.

First of all, the four-factor IPCA model that implies time-varying loadings successfully describes the riskiness of the currency returns and the risk compensation. We have shown that this model explains 96,97% of currency return variation for managed portfolios and 64,12% for individual currencies. Furthermore, most of the model's predictivity aligns with the instrumented exposures to the risk factors, not intercepts. We have investigated that six characteristics out of the initial 16 characteristics are statistically significant at a 5% level in the IPCA model and considerably contribute to the IPCA model performance.

Secondly, the IPCA factors that explain the cross-section of currency returns are related to the common risk factors in the forex market. These are global volatility, carry trade, dollar, and momentum factors.

Finally, we have compared the performance of the IPCA model with the existing observable factors models and proved that the IPCA outperforms both in-sample and out-of-sample. Nevertheless, the models with pre-specified observable factors still show outstanding performance and explain the majority of currency returns' variation. This means that the time-varying exposures in the IPCA approach enhanced the ability of factors to describe the relationship between risk and return in exchange rate markets.

7.1 Further recommendations

In this research, we have shown the statistical significance and efficiency of the IPCA model with latent factors for currency markets. Despite this fact, there is still a need for portfolio managers to evaluate the economic significance and practical performance of the derived factors. That is, portfolio managers can apply the IPCA model results to predict the currency fluctuation and use the predictions as a benchmark in their currency portfolio construction. The economic significance will be proved if the predicted currency fluctuations coincide with the actual exchange rate in the predicted period and yield positive returns on the investors' portfolio.

Appendix A

Data range of the FX data

TABLE A.1: Date range of the available FX data.

Appendix B

IPCA factors summary statistics

TABLE B.1: Summary statistics of the IPCA factors

The table provides summary statistics (mean, standard deviation, Sharpe ratio, kurtosis, skewness) for each IPCA factor on the yearly basis. The IPCA factors data is available from January 1992 to September 2018.

Appendix C

IPCA and observable factors correlation

TABLE C.1: Correlation between the IPCA and observable factors The table provides an overview of the pairwise correlation between IPCA and observable factors.

Appendix D

Γ*β* **coefficient estimates**

FIGURE D.1: Γ*^β* coefficient estimates This figure shows each column of the Γ*^β* matrix with the characteristics' weights for the IPCA configuration with $K = 4$.

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